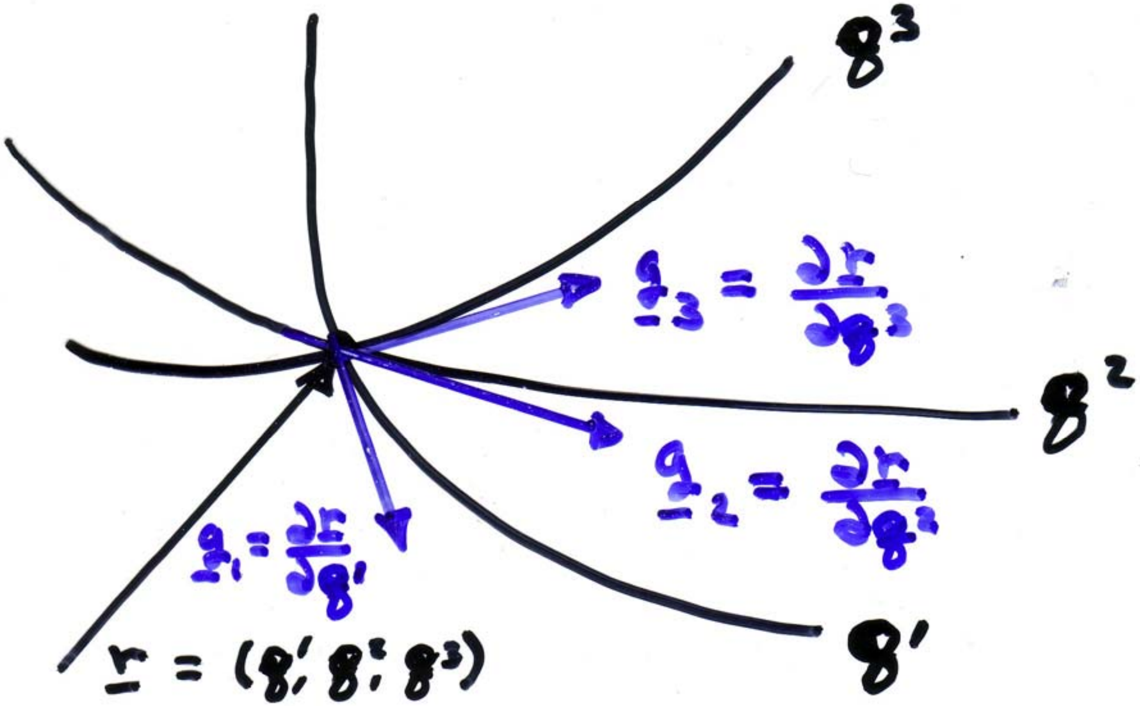
A.2 Curvilinear coordinates systems



Chain Rule

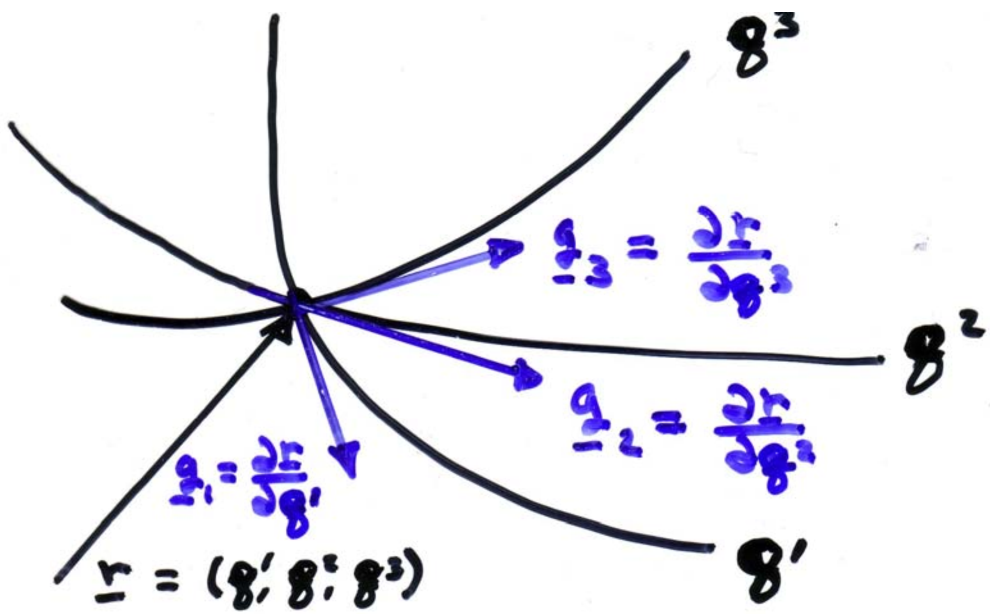
$$d\underline{r} = \underbrace{\frac{\partial \underline{r}}{\partial g^1}}_{\underline{g}_1} dg^1 + \underbrace{\frac{\partial \underline{r}}{\partial g^2}}_{\underline{g}_2} dg^2 + \underbrace{\frac{\partial \underline{r}}{\partial g^3}}_{\underline{g}_3} dg^3$$

Base Vectors

$$\boxed{\underline{g}_i = \frac{\partial \underline{r}}{\partial \theta^i}} = \frac{\partial x^p}{\partial \theta^i}\,\underline{e}_p \qquad (A2\text{-}4)$$

$$d\underline{r} = \underline{g}_1\, d\theta^1 + \underline{g}_2\, d\theta^2 + \underline{g}_3\, d\theta^3 = \underline{g}_i\, d\theta^i$$

(Convention de $\sum_i$ utilisée)



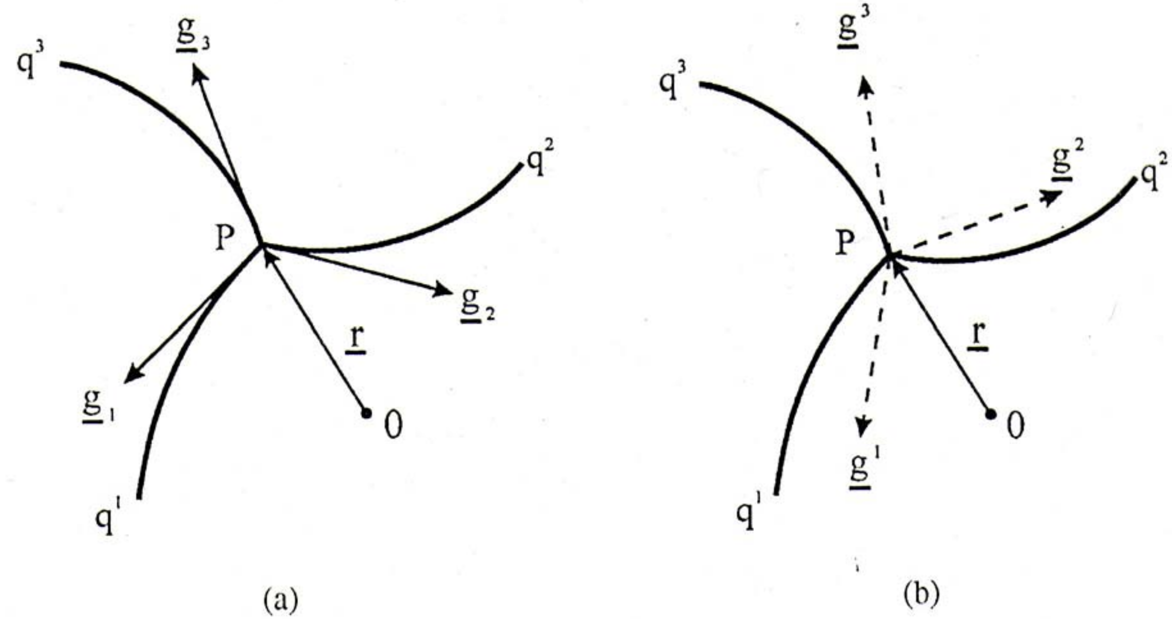


Figure A.2-2 Generalized coordinate system and base vectors: (a) covariant (tangential) base vectors; (b) contravariant (normal) base vectors

$$\boxed{\underline{g}^n = \underline{\nabla} q^n = \text{grad } q^n = \frac{\partial q^n}{\partial x^l} \underline{\delta}_l,} \tag{A.2-5a}$$

where the operator $\underline{\nabla}$ is defined by

$$\underline{\nabla} \equiv \underline{g}^n \frac{\partial}{\partial q^n}. \tag{A.2-5b}$$

Note that neither $\underline{g}_m$ nor $\underline{g}^n$ are necessarily unit vectors, but from orthogonality it follows that

$$\underline{g}_m \cdot \underline{g}^n = \delta^n_m \tag{A.2-6a}$$

Example A.2-1 Base Vectors in Spherical Coordinates

Soit (x^1, x^2, x^3) coordonnées Cartésiennes

et (r, θ, ϕ) coordonnées sphériques

From basic trigonometry

$$x^1 = r\sin\theta\cos\phi \, , \quad x^2 = r\sin\theta\sin\phi \, , \quad x^3 = r\cos\theta$$

$$\underline{g}_r = \frac{\partial x^1}{\partial r}\underline{S}_1 + \frac{\partial x^2}{\partial r}\underline{S}_2 + \frac{\partial x^3}{\partial r}\underline{S}_3$$

$$= \sin\theta\cos\phi\,\underline{S}_1 + \sin\theta\sin\phi\,\underline{S}_2 + \cos\theta\,\underline{S}_3$$

$$\underline{g}_\theta = \frac{\partial x^1}{\partial \theta}\underline{S}_1 + \frac{\partial x^2}{\partial \theta}\underline{S}_2 + \frac{\partial x^3}{\partial \theta}\underline{S}_3$$

$$= r\cos\theta\cos\phi\,\underline{S}_1 + r\cos\theta\sin\phi\,\underline{S}_2 - r\sin\theta\,\underline{S}_3$$

$$\underline{g}_\phi = \frac{\partial x^1}{\partial \phi}\underline{S}_1 + \frac{\partial x^2}{\partial \phi}\underline{S}_2 + \frac{\partial x^3}{\partial \phi}\underline{S}_3$$

$$= -r\sin\theta\sin\phi\,\underline{S}_1 + r\sin\theta\cos\phi\,\underline{S}_2$$

From B.S.L. A.6

$$\underline{\underline{\delta}}_r = \sin\theta\cos\phi\,\underline{\underline{\delta}}_1 + \sin\theta\sin\phi\,\underline{\underline{\delta}}_2 + \cos\theta\,\underline{\underline{\delta}}_3$$

$$\underline{\underline{\delta}}_\theta = \cos\theta\cos\phi\,\underline{\underline{\delta}}_1 + \cos\theta\sin\phi\,\underline{\underline{\delta}}_2 - \sin\theta\,\underline{\underline{\delta}}_3$$

$$\underline{\underline{\delta}}_\phi = -\sin\phi\,\underline{\underline{\delta}}_1 + \cos\phi\,\underline{\underline{\delta}}_2$$

$$\left.\begin{aligned} \underline{g}_r &= \underline{\underline{\delta}}_r \\ \underline{g}_\theta &= r\,\underline{\underline{\delta}}_\theta \\ \underline{g}_\phi &= r\sin\theta\,\underline{\underline{\delta}}_\phi \end{aligned}\right\}$$

Not necessarily of unit value and dimensionless!

To obtain the reciprocal (contravariant) base vectors

$$\underline{g}^i \cdot \underline{g}_j = \delta^i_j$$

$$\Rightarrow \quad \underline{g}^r = \underline{e}_r$$

$$\underline{g}^\theta = \frac{\underline{e}_\theta}{r}$$

$$\underline{g}^\phi = \frac{1}{r\sin\theta}\underline{e}_\phi$$

A.2-2 Transformation rule for Vectors

$$q^m = q^m(\bar{q}^n)$$

or $$\bar{q}^n = \bar{q}^n(q^m)$$

at $$\bar{\underline{g}}_m = \frac{\partial \underline{r}}{\partial \bar{q}^m} \quad \text{at} \quad \bar{\underline{g}}^n = \frac{\partial \bar{q}^n}{\partial x^l}\underline{e}_l = \nabla \bar{q}^n$$

For any vector

$$\underline{v} = v^m \underline{g}_m = \bar{v}^n \bar{\underline{g}}_n$$

$$\underline{g}_m = \underbrace{\left(\frac{\partial \underline{r}}{\partial \bar{q}^n}\right)}_{\bar{\underline{g}}_n}\left(\frac{\partial \bar{q}^n}{\partial q^m}\right) \longleftarrow \text{Chain Rule}$$

$$\Rightarrow \quad \underline{v} = v^m \bar{\underline{g}}_n \left(\frac{\partial \bar{q}^n}{\partial q^m}\right) = \bar{v}^n \bar{\underline{g}}_n$$

$$\Rightarrow \quad \boxed{\bar{v}^n = v^m \left(\frac{\partial \bar{q}^n}{\partial q^m}\right)}$$

Tranformation of a contravariant component

N.B. For the coordinates:

$$\bar{q}^i = \bar{q}^i(q^1, q^2, q^3)$$

The Chain Rule:

$$d\bar{q}^i = \underbrace{\frac{\partial \bar{q}^i}{\partial q^j}\, dq^j}$$

Contravariant transformation

② Pour les vecteurs de base :

$$\underline{g}_m = \frac{\partial \underline{r}}{\partial \bar{\xi}^n}\,\frac{\partial \bar{\xi}^n}{\partial \xi^m}$$

$$= \bar{\underline{g}}_n\,\frac{\partial \bar{\xi}^n}{\partial \xi^m}$$

} transformation d'une "composante" covariante

<u>Composante covariante</u> :

$$\underline{V} = V_m\,\underline{g}^m = \bar{V}_n\,\bar{\underline{g}}^n$$

$$\nabla \xi^m = \frac{\partial \xi^m}{\partial \bar{\xi}^n}\,\underbrace{\nabla \bar{\xi}^n}_{\bar{\underline{g}}^n}$$

$$\Rightarrow \quad V_m\,\frac{\partial \xi^m}{\partial \bar{\xi}^n}\,\bar{\underline{g}}^n = \bar{V}_n\,\bar{\underline{g}}^n$$

et

$$\boxed{\bar{V}_n = V_m\,\frac{\partial \xi^m}{\partial \bar{\xi}^n}}$$

Transformation d'une <u>composante covariante</u>

N.B. Transformation of reciprocal base vector

$$\underline{g}^m = \nabla z^m = \frac{\partial z^m}{\partial \bar{z}^n} \nabla \bar{z}^n = \frac{\partial z^m}{\partial \bar{z}^n} \bar{\underline{g}}^n$$

$$\boxed{\underline{g}^m = \frac{\partial z^m}{\partial \bar{z}^n} \bar{\underline{g}}^n}$$

si trasforma come una componente contravariante

Transformation of second order tensor

$$\tau^{ij} = \left(\frac{\partial z^i}{\partial \bar{z}^k}\right)\left(\frac{\partial z^j}{\partial \bar{z}^l}\right) \bar{\tau}^{kl}$$

Componenti contravarianti

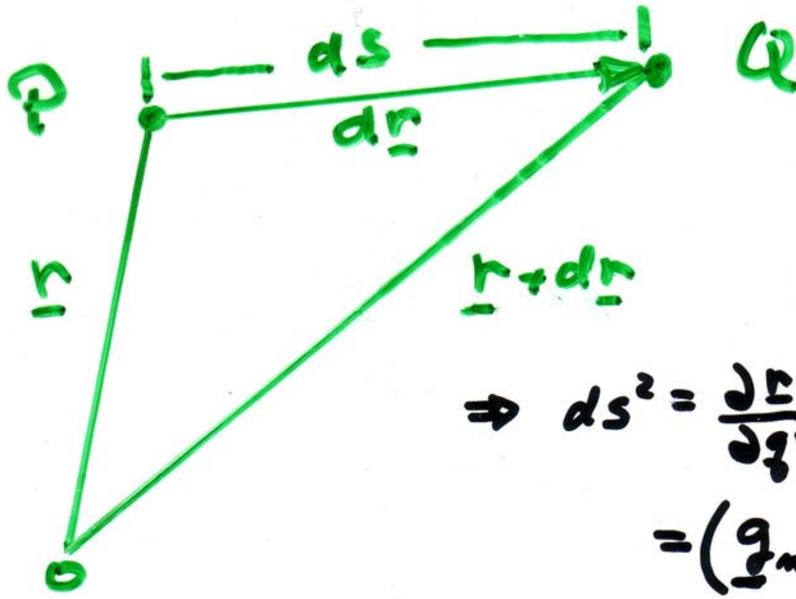
$$\tau_{ij} = \left(\frac{\partial \bar{z}^k}{\partial z^i}\right)\left(\frac{\partial \bar{z}^l}{\partial z^j}\right) \bar{\tau}_{kl}$$

componenti covarianti

$$\tau^i_{\ j} = \left(\frac{\partial z^i}{\partial \bar{z}^k}\right)\left(\frac{\partial \bar{z}^l}{\partial x^j}\right) \bar{\tau}^k_{\ l}$$

componenti miste.

A.2-4 Metric Tensor



$$ds^2 = d\mathbf{r}\cdot d\mathbf{r}$$

$$\Rightarrow ds^2 = \frac{\partial \mathbf{r}}{\partial \xi^m} d\xi^m \cdot \frac{\partial \mathbf{r}}{\partial \xi^n} d\xi^n$$

$$= \underbrace{\left(\mathbf{g}_m \cdot \mathbf{g}_n\right)}_{g_{mn}} d\xi^m d\xi^n$$

$$\Rightarrow \quad ds^2 = \underbrace{g_{mn}}\, d\xi^m d\xi^n$$

Covariant components of metric tensor

Properties:

N.B. ① en coord. orthonormales $g_{mn} = \delta_{mn}$

en général $g_{mn} \neq \delta_{mn}$

② $g_{mn} = g_{nm}$ (symétrique)

③ $\underline{g}^r \cdot \underline{g}^s = g^{rs}$ composantes contravariantes du tenseur métrique.

④ $$g_{mj}\, g^{jn} = (\underline{g}_m \cdot \underline{g}_j)(\underline{g}^j \cdot \underline{g}^n)$$
$$= (\underline{g}_m \cdot \underline{g}^j)(\underline{g}_j \cdot \underline{g}^n)$$
$$= \delta_m^{\,j}\, \delta_j^{\,n} = \delta_m^{\,n}$$

A.2-5 Physical Components

On a: $\underline{v} = v_m \, \underline{g}^m = v^n \, \underline{g}_n$

$\cdot \underline{g}_r$ $\quad v_m \, \underline{g}^m \cdot \underline{g}_r = v^n \, \underline{g}_n \cdot \underline{g}_r$

$$v_m \, \delta^m_r = v^n \, g_{nr}$$

$$\Rightarrow \underbrace{\boxed{v_r = g_{nr} \, v^n}}_{\Sigma_n}$$

Transformation de Comp. contrav. à Comp. covar.

Physical component of the base vectors

$$\boxed{g_{(n)} = \frac{\underline{g}_n}{\left|\underline{g}_n\right|} = \frac{\underline{g}_n}{\sqrt{g_{nn}}}}$$

No summation!

Physical components of a vector

$$v_{(n)} = \sqrt{g_{nn}}\, v^{n} = \sqrt{g_{nn}}\, g^{nm} v_{m} = \sqrt{g^{nn}}\, v_{n}$$

Physical components of a second order tensor

$$T_{(mr)} = \sqrt{g_{mm}}\sqrt{g_{rr}}\, T^{mr} = \sqrt{g^{mm}}\sqrt{g^{rr}}\, T_{mr} = \sqrt{g^{mm}}\sqrt{g_{rr}}\, T_{m}^{\ r}$$

Etc…

Examples A.2-4 & 5: metric tensor and physical components in spherical coordinates

$$g_{ij} = \underline{g}_i \cdot \underline{g}_j \qquad g^{ij} = \underline{g}^i \cdot \underline{g}^j$$

$$g_{rr} = \underline{\delta}_r \cdot \underline{\delta}_r = 1 = g^{rr}$$

$$g_{\theta\theta} = r\underline{\delta}_\theta \cdot r\underline{\delta}_\theta = r^2 = (1/g^{\theta\theta})^{\cancel{2}}$$

$$g_{\phi\phi} = r\sin\theta\,\underline{\delta}_\phi \cdot r\sin\theta\,\underline{\delta}_\phi = r^2\sin^2\theta = (1/g^{\phi\phi})^{\cancel{2}}$$

$$g_{ij} = 0 \text{ for } i \neq j \qquad 0 = g^{ij} \quad j \neq i$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{pmatrix}$$

Vector

$$\underline{V} = V_r \underline{e}_r + V_\theta \underline{e}_\theta + V_\phi \underline{e}_\phi$$

$$= V_1 \underline{g}^1 + V_2 \underline{g}^2 + V_3 \underline{g}^3$$

$$= V^1 \underline{g}_1 + V^2 \underline{g}_2 + V^3 \underline{g}_3$$

$$\Rightarrow V(1) = \sqrt{g_{11}}\, V^1 = \sqrt{g^{11}}\, V_1 = V_r = V_1 = V^1$$

$$V(2) = \sqrt{g_{22}}\, V^2 = \sqrt{g^{22}}\, V_2 = V_\theta$$

$$V^2 = V_\theta / r \quad \text{et} \quad V_2 = r V_\theta$$

$$V(3) = \sqrt{g_{33}}\, V^3 = \sqrt{g^{33}}\, V_3 = V_\phi$$

$$V^3 = \frac{V_\phi}{r \sin\theta} \quad \text{et} \quad V_3 = r \sin\theta\, V_\phi$$

A.3 Covariant differentiation

Operator DEL, ∇ :

$$\nabla \equiv \underline{g}^i \frac{\partial}{\partial \xi^i}$$

Gradient of a scalar

$$\boxed{\nabla s = \underline{g}^i \frac{\partial s}{\partial \xi^i} = \underline{g}^i s_{,i}}$$

Gradient of a vector

$$\nabla \underline{v} = \underline{g}^i \frac{\partial}{\partial \xi^i}\left(\underline{g}^j v_j\right)$$

$$= \underline{g}^i \underline{g}^j \frac{\partial v_j}{\partial \xi^i} + \underline{g}^i v_j \underbrace{\frac{\partial}{\partial \xi^i} \underline{g}^j}_{-\Gamma^j_{ki}\, \underline{g}^k}$$

Christoffel symbols

Christoffel symbols

$$\frac{\partial \underline{g}_i}{\partial \theta^j} = \Gamma^k_{ij}\,\underline{g}_k = \left\{ {k \atop ij} \right\} \underline{g}_k$$

$$\frac{\partial \underline{g}^i}{\partial \theta^j} = -\Gamma^i_{kj}\,\underline{g}^k$$

They express
the dependence
of base vectors
on coordinates.

Properties:

$$\Gamma^{l}_{mj} = \frac{1}{2} g^{lk} \left[\frac{\partial g_{jk}}{\partial q^{m}} + \frac{\partial g_{mk}}{\partial q^{j}} - \frac{\partial g_{mj}}{\partial q^{k}} \right] \quad \text{(A.3-10)}$$

$$\Gamma^{l}_{mj} = \Gamma^{l}_{jm} \quad \text{symétrique}$$

$$\nabla \underline{v} = \underline{g}^{i} \underline{g}^{j} \frac{\partial}{\partial q^{i}} v_{j} - \underline{g}^{i} \underline{g}^{k} v_{j} \Gamma^{j}_{ki}$$

$$= \underline{g}^{i} \underline{g}^{j} \left(\frac{\partial v_{j}}{\partial q^{i}} - v_{k} \Gamma^{k}_{ji} \right)$$

$$\Rightarrow \boxed{\nabla \underline{v} = \underline{g}^{i} \underline{g}^{j} \, v_{j,i}}$$

Also:

$$\nabla \underline{v} = \underline{g}^i \underline{g}_j \, v^j_{\,,i}$$

$$\nabla \underline{\underline{\tau}} = \underline{g}^i \underline{g}^j \underline{g}^k \, \tau_{jk,i}$$

$$\tau_{jk,i} = \frac{\partial}{\partial q^i} \tau_{jk} - \Gamma^{l}_{ji} \tau_{lk} - \Gamma^{l}_{ki} \tau_{jl}$$

Divergence of a vector:

$$\underline{\nabla} \cdot \underline{v} = \underline{g}^i \frac{\partial}{\partial q^i} \cdot v_j \underline{g}^j = g^{ij} \frac{\partial v_j}{\partial q^i} - g^{ik} v_j \Gamma^j_{ki}$$

$$= g^{ij} \left(\frac{\partial v_j}{\partial q^i} - v_k \Gamma^k_{ji} \right) = g^{ij} v_{j,i}$$

$$\Rightarrow \underline{\nabla} \cdot \underline{v} = v^i_{\;,i} \text{ a scalar}$$